

Supermodular ordering of binomial, Poisson and Gaussian random vectors by tree-based correlations

Bünyamin Kızıldemir* Nicolas Privault†

Division of Mathematical Sciences

School of Physical and Mathematical Sciences

Nanyang Technological University

637371 Singapore

May 17, 2016

Abstract

We construct a tree-based dependence structure for the representation of binomial, Poisson and Gaussian random vectors having a given covariance matrix, using sums of independent random variables. This construction allows us to characterize the supermodular ordering of such random vectors via the componentwise ordering of their covariance matrices. Our method relies on the representation of dependent components using binary trees on the discrete d -dimensional hypercube C_d , and on Möbius inversion techniques. In the case of Poisson random vectors this approach involves Lévy measures on C_d , and it is consistent with the approximation of Poisson and multivariate Gaussian random vectors by binomial vectors.

Key words: Stochastic ordering; supermodular functions; binary trees, Möbius inversion, Poisson random vectors, binomial random vectors.

Mathematics Subject Classification: 60E15; 62H20; 60E07, 05C05; 06A11.

1 Introduction

A d -dimensional random vector $X = (X_1, \dots, X_d)$ is said to be dominated by another random vector $Y = (Y_1, \dots, Y_d)$ in the supermodular order, and one writes $X \leq_{\text{sm}} Y$,

*bunjamin001@e.ntu.edu.sg

†nprivault@ntu.edu.sg

if

$$E[\Phi(X)] \leq E[\Phi(Y)],$$

for all sufficiently integrable *supermodular* functions, i.e. for all functions $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$\Phi(x) + \Phi(y) \leq \Phi(x \wedge y) + \Phi(x \vee y), \quad x, y \in \mathbb{R}^d,$$

where the maximum \vee and the minimum \wedge are defined with respect to the componentwise order of $x, y \in \mathbb{R}^d$. The supermodular stochastic ordering is used in particular to capture a preference for greater inter-dependence in economic variables.

In the case where X and Y are multivariate Gaussian vectors, supermodular ordering has been characterized by the componentwise ordering of their covariance matrices in [5] Theorem 4.2, cf. also Theorem 3.13.5 of [6]. In [4], a dependence structure under which supermodular ordering can be characterized under the same covariance condition has been provided for Poisson random vectors, based on a decomposition of their Lévy measures on the vertices of the d -dimensional unit hypercube C_d .

In this paper we construct a general dependence structure for binomial and Poisson random vectors, under which the supermodular ordering can be characterized by the ordering of covariance matrices. This approach extends the results of [4] to a larger family of dependence structures, based on an arrangement of random variables according to a binary tree and on Möbius inversion. In the Gaussian case it allows us to represent any multivariate Gaussian vector using sums of independent Gaussian random variables as in (2.5) below, and to recover the result of [5] as a consequence. Similarly, in the binomial, gamma and Poisson settings it yields a construction of random vectors having an arbitrarily given covariance matrix, and it provides the associated characterization of binomial and Poisson supermodular ordering, cf. Theorems 3.1 and 4.2. We refer the reader to [2] and references therein for the use of a different type of tree-based dependence in the setting of Bernoulli random vectors.

We proceed as follows. In Section 2 we construct a general dependence structure that uses binary trees on the vertices of the d -dimensional hypercube. In Section 3 we apply this dependence structure to the characterization of the binomial supermodular

ordering via the componentwise ordering of covariances, cf. Theorem 3.1. In Section 4 we deal with the case of Poisson random vectors via the use of Lévy measures on the vertices of the d -dimensional unit hypercube C_d , cf. Theorem 4.2. This also includes extensions to the increasing supermodular order, cf. Proposition 4.3. This result naturally extends to the supermodular ordering of sums of binomial, multivariate Gaussian and Poisson random vectors. We also include a remark on the related convex ordering problem for Poisson random vectors in Proposition 4.4.

2 Tree-based correlation structures

In this section we introduce the general dependence structure used in this paper. Let (e_1, \dots, e_d) denote the canonical basis of \mathbb{R}^d , and let

$$C_d := \{0, 1\}^d = \{x = (x_1, \dots, x_d) : x_i \in \{0, 1\}, i = 1, \dots, d\}$$

denote the set of vertices of the d -dimensional unit hypercube.

We identify C_d to the power set $\{0, 1\}^d \simeq \{S \in \{1, \dots, d\}\}$ of $\{1, \dots, d\}$, i.e. each $x_S = (x_1, \dots, x_d) \in C_d$ is identified to its index set $S = \{i \in \{1, \dots, d\} : x_i = 1\}$. In particular, we write $a \in x = (x_1, \dots, x_d)$ when $x_a = 1$, and $x \setminus \{a\}$ for $(x_i \mathbf{1}_{\{i \neq a\}})_{i=1, \dots, d}$.

We also endow C_d with the natural inclusion ordering of index sets i.e. we write $x \preceq y$ when $x \subseteq y$, or equivalently $0 \leq x_i \leq y_i \leq 1$, $i = 1, \dots, d$, and $x \prec y$ when $x \preceq y$ and $x \neq y$, i.e. $x \subsetneq y$.

Random vectors

Given $(e_{k,l})_{1 \leq k \leq l \leq d}$ a family of elements of C_d with $e_{k,k} := e_k$, $k = 1, \dots, d$, and a family $(X_{i,j})_{1 \leq i \leq j \leq d}$ of independent random variables, consider the random vector $X = (X_1, \dots, X_d)$ given by

$$X_i := \sum_{i \in e_{k,l}} X_{k,l}, \quad i = 1, \dots, d.$$

In other words, we have

$$X = \sum_{i=1}^d e_i \sum_{i \in e_{k,l}} X_{k,l} = \sum_{1 \leq k \leq l \leq d} X_{k,l} e_{k,l}.$$

Denoting by $(m_{k,l})_{1 \leq k \leq l \leq d}$ and $(\sigma_{k,l}^2)_{1 \leq k \leq l \leq d}$ the respective means and variances of $(X_{k,l})_{1 \leq k \leq l \leq d}$, we have

$$E[X_i] = \sum_{i \in e_{k,l}} m_{k,l}, \quad i = 1, \dots, d,$$

and

$$\text{Cov}(X_i, X_j) = \sum_{i,j \in e_{k,l}} \text{Var}[X_{k,l}] = \sum_{e_{i,j} \preceq e_{k,l}} \sigma_{k,l}^2, \quad 1 \leq i \leq j \leq d. \quad (2.1)$$

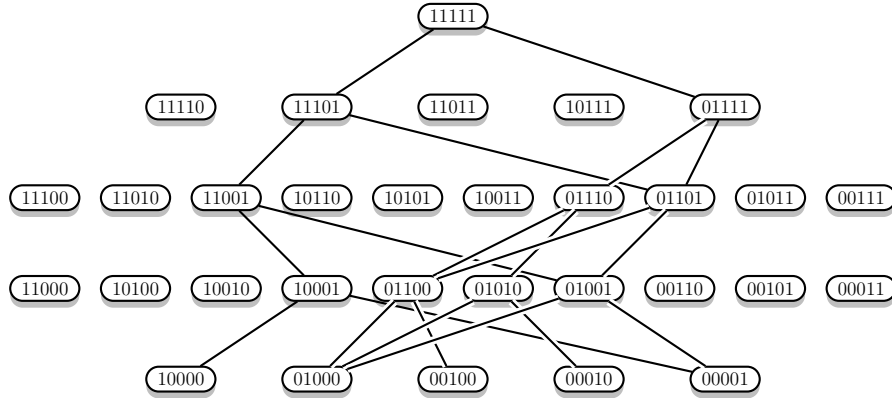
Binary trees

From now on we work under the following assumption on $(e_{k,l})_{1 \leq k < l \leq d}$.

(H) The family $(e_{k,l})_{1 \leq k < l \leq d} \subset C_d$ forms a binary tree of size $d(d+1)/2$ in which every node $e_{k,l}$ has two children $e_{k,l \setminus \{k\}}$ and $e_{k,l \setminus \{l\}}$.

In particular, $(e_{k,l})_{1 \leq k < l \leq d}$ forms a binary tree with height at most d for the partial order \preceq .

Example. When $d = 5$, consider



with

$$\begin{aligned}
e_{1,2} &= \begin{vmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix} \\
e_{1,3} &= \\
e_{1,4} &= \\
e_{1,5} &= \\
e_{2,3} &= \\
e_{2,4} &= \\
e_{2,5} &= \\
e_{3,4} &= \\
e_{3,5} &= \\
e_{4,5} &=
\end{aligned}$$

and

$$\begin{cases}
X_1 = X_{1,1} + X_{1,2} + X_{1,3} + X_{1,4} + X_{1,5} \\
X_2 = X_{2,2} + X_{1,2} + X_{1,3} + X_{1,4} + X_{2,3} + X_{2,4} + X_{2,5} + X_{3,4} + X_{3,5} + X_{4,5} \\
X_3 = X_{3,3} + X_{1,3} + X_{1,4} + X_{2,3} + X_{3,4} + X_{3,5} + X_{4,5} \\
X_4 = X_{4,4} + X_{1,4} + X_{2,4} + X_{3,4} + X_{4,5} \\
X_5 = X_{5,5} + X_{1,2} + X_{1,3} + X_{1,4} + X_{1,5} + X_{2,5} + X_{3,5} + X_{4,5}.
\end{cases}$$

Möbius inversion

By Möbius inversion, cf. Proposition 2 of [8] or Proposition 2.6.3 of [7], the coefficients $(\sigma_{k,l}^2)_{1 \leq k \leq l \leq d}$ in (2.1) can be recovered using the covariances $(\text{Cov}(X_i, X_j))_{1 \leq i \leq j \leq d}$ as

$$\sigma_{k,l}^2 = \sum_{e_{k,l} \preceq e_{i,j}} \mu(e_{i,j}, e_{k,l}) \text{Cov}(X_i, X_j), \quad 1 \leq k \leq l \leq d, \quad (2.2)$$

where $\mu(x, y)$ is the Möbius function defined recursively by $\mu(x, x) := 1$ and

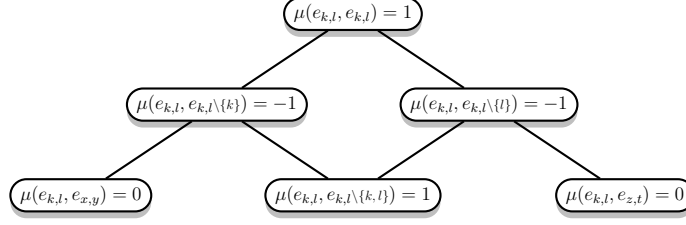
$$\mu(x, y) = - \sum_{y \prec z \preceq x} \mu(x, z), \quad x, y \in C_d, \quad (2.3)$$

cf. Proposition 1 of [8]. Given $e_{k,l} \in C_d$, the two children $e_{k,l \setminus \{k\}}$, and $e_{k,l \setminus \{l\}}$ of $e_{k,l}$ have themselves a unique common child $e_{k,l \setminus \{k,l\}}$, and (2.3) yields

$$\begin{cases} \mu(e_{k,l}, e_{k,l}) = 1, \end{cases} \quad (2.4a)$$

$$\begin{cases} \mu(e_{k,l}, e_{k,l \setminus \{k\}}) = -1, \\ \mu(e_{k,l}, e_{k,l \setminus \{l\}}) = -1, \\ \mu(e_{k,l}, e_{k,l \setminus \{k,l\}}) = 1, \end{cases} \quad (2.4b)$$

as shown in the next graph:



In addition, we have $\mu(e_{k,l}, e_{i,j}) = 0$ in all other cases.

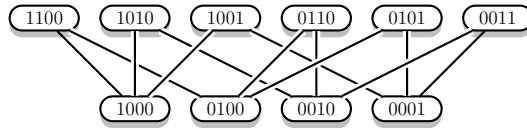
In other words, (2.1) can be solved recursively for $(\sigma_{k,l}^2)_{1 \leq k \leq l \leq d}$ given the data of $(\text{Cov}(X_i, X_j))_{1 \leq i \leq j \leq d}$ by starting from the equality $\text{Cov}(X_k, X_l) = \sigma_{k,l}^2$ at a root $e_{k,l}$ of the tree, and then by moving down the tree step by step until each leaf e_i .

Examples. *i)* Comonotonic vectors. The binary tree contains the node $e_{k,l} = \overline{111 \dots 11}$ as root, and all coefficients $\sigma_{i,j}^2$ vanish for $(i,j) \neq (k,l)$, which corresponds to the vector $(X_{k,l}, X_{k,l}, \dots, X_{k,l})$ with $\sigma_{k,l}^2 = \text{Var}[X_{k,l}]$.

ii) Pairwise dependence. The binary tree is reduced to the d leaves e_1, \dots, e_d , and to their parents

$$e_{k,l} = (0, \dots, 0, \underset{\uparrow k}{1}, 0, \dots, 0, \underset{\uparrow l}{1}, 0, \dots, 0), \quad 1 \leq k \leq l \leq d,$$

as in the following example with $d = 4$:



Here, the vector $(X_i)_{i=1, \dots, d}$ is given by

$$\begin{cases} X_1 = X_{1,1} + X_{1,2} + X_{1,3} + X_{1,4} \\ X_2 = X_{1,2} + X_{2,2} + X_{2,3} + X_{2,4} \\ X_3 = X_{1,3} + X_{2,3} + X_{3,3} + X_{3,4} \\ X_4 = X_{1,4} + X_{2,4} + X_{3,4} + X_{4,4}, \end{cases}$$

and for any $d \geq 1$, by (2.1) we have

$$\text{Cov}(X_i, X_j) = \sigma_{i,j}^2, \quad 1 \leq i < j \leq d,$$

and

$$\text{Var}[X_i] = \sum_{j=1}^{i-1} \sigma_{j,i}^2 + \sum_{j=i}^d \sigma_{i,j}^2, \quad i = 1, \dots, d.$$

iii) Gaussian vectors. If (U_1, \dots, U_d) is a centered multivariate Gaussian random vector with covariance matrix $(\text{Cov}(U_i, U_j))_{1 \leq i \leq j \leq d}$ we can apply the Möbius inversion (2.2) in order to determine the variance coefficients $(\sigma_{k,l}^2)_{1 \leq k \leq l \leq d} = (\text{Var}[Z_{k,l}])_{1 \leq k \leq l \leq d}$ of independent centered Gaussian random variables such that

$$U_i := \sum_{i \in e_{k,l}} Z_{k,l}, \quad i = 1, \dots, d. \quad (2.5)$$

iv) As in (iii) above, binomial, Poisson and gamma random vectors having a given covariance matrix can be constructed by solving (2.2) on a binary tree since those distributions are characterized by their variance parameters and they are stable by summation for a given scale parameter. However in this case the construction may not be unique depending on the chosen binary tree, as their joint distribution is not characterized by their covariance matrices.

v) The dependence structure considered in [4] for Poisson random vectors corresponds to the binary tree built on the $d(d-1)/2$ nodes

$$e_{i,j} = (1, \dots, 1, \underset{i}{\underset{\uparrow}{1}}, 0, \dots, 0, \underset{j}{\underset{\uparrow}{1}}, 0, \dots, 0), \quad 1 \leq i < j \leq d,$$

and on the d leaves e_1, \dots, e_d .

3 Binomial random vectors

Consider (Z_1, \dots, Z_n) independent Bernoulli random variables with parameter $p \in [0, 1]$ and $(A(e_{k,l}))_{1 \leq k \leq l \leq d}$ a *partition* of $\{1, \dots, n\}$. Let $(X_{A(e_{k,l})})_{1 \leq k \leq l \leq d}$ denote the family of independent binomial random variables given by

$$X_{A(e_{k,l})} := \sum_{i \in A(e_{k,l})} Z_i, \quad 1 \leq k \leq l \leq d,$$

with

$$m_{k,l} = E[X_{A(e_{k,l})}] = p|A(e_{k,l})| \quad 1 \leq k \leq l \leq d,$$

where $|A(e_{k,l})|$ denotes the cardinality of $A(e_{k,l})$, and

$$\sigma_{k,l}^2 = \text{Var}[X_{A(e_{k,l})}] = pq|A(e_{k,l})|, \quad 1 \leq k \leq l \leq d,$$

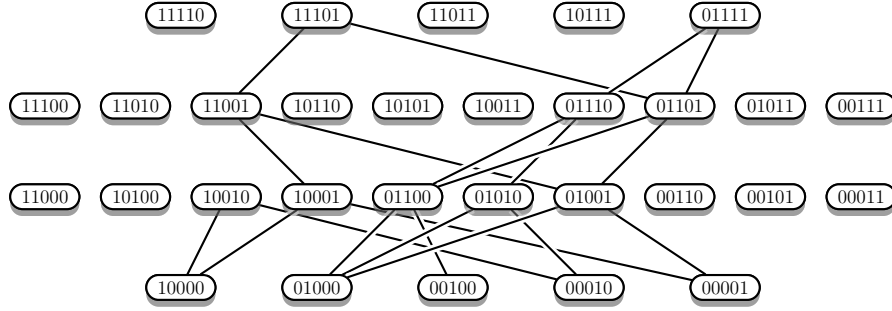
where $q := 1 - p$. Let now

$$A_i := \bigcup_{i \in e_{k,l}} A(e_{k,l}), \quad i = 1, \dots, d,$$

and consider the vector $(X_{A_1}, \dots, X_{A_d})$ of binomial random variables defined by

$$X_{A_i} := \sum_{k \in A_i} Z_k = \sum_{i \in e_{k,l}} X_{A(e_{k,l})}, \quad i = 1, \dots, d. \quad (3.1)$$

Example. When $d = 5$, the binary tree



corresponds to

$$\begin{aligned} e_{S(1,2)} &= \begin{vmatrix} 1 & 1 & 0 & 0 & 1 \end{vmatrix} \\ e_{S(1,3)} &= \begin{vmatrix} 1 & 1 & 1 & 0 & 1 \end{vmatrix} \\ e_{S(1,4)} &= \begin{vmatrix} 1 & 0 & 0 & 1 & 0 \end{vmatrix} \\ e_{S(1,5)} &= \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \end{vmatrix} \\ e_{S(2,3)} &= \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \end{vmatrix} \\ e_{S(2,4)} &= \begin{vmatrix} 0 & 1 & 0 & 1 & 0 \end{vmatrix} \\ e_{S(2,5)} &= \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \end{vmatrix} \\ e_{S(3,4)} &= \begin{vmatrix} 0 & 1 & 1 & 1 & 0 \end{vmatrix} \\ e_{S(3,5)} &= \begin{vmatrix} 0 & 1 & 1 & 0 & 1 \end{vmatrix} \\ e_{S(4,5)} &= \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \end{vmatrix} \end{aligned}$$

and

$$\begin{cases} X_{A_1} = X_{A(e_{1,1})} + X_{A(e_{1,2})} + X_{A(e_{1,3})} + X_{A(e_{1,4})} + X_{A(e_{1,5})} \\ X_{A_2} = X_{A(e_{2,2})} + X_{A(e_{1,2})} + X_{A(e_{1,3})} + X_{A(e_{2,3})} + X_{A(e_{2,4})} + X_{A(e_{2,5})} + X_{A(e_{3,4})} + X_{A(e_{3,5})} + X_{A(e_{4,5})} \\ X_{A_3} = X_{A(e_{3,3})} + X_{A(e_{1,3})} + X_{A(e_{2,3})} + X_{A(e_{3,4})} + X_{A(e_{3,5})} + X_{A(e_{4,5})} \\ X_{A_4} = X_{A(e_{4,4})} + X_{A(e_{1,4})} + X_{A(e_{2,4})} + X_{A(e_{3,4})} + X_{A(e_{4,5})} \\ X_{A_5} = X_{A(e_{5,5})} + X_{A(e_{1,5})} + X_{A(e_{1,3})} + X_{A(e_{1,5})} + X_{A(e_{2,5})} + X_{A(e_{3,5})} + X_{A(e_{4,5})}. \end{cases}$$

In general we have

$$E[X_{A_i}] = p \sum_{i \in e_{k,l}} |A(e_{k,l})|, \quad i = 1, \dots, d,$$

and

$$\text{Cov}(X_{A_i}, X_{A_j}) = pq \sum_{e_{i,j} \preceq e_{k,l}} |A(e_{k,l})|, \quad 1 \leq i \leq j \leq d.$$

with the inversion formula

$$pq|A(e_{k,l})| = \sum_{e_{k,l} \preceq e_{i,j}} \mu(e_{i,j}, e_{k,l}) \text{Cov}(X_i, X_j), \quad 1 \leq k \leq l \leq d,$$

that follows from (2.2). Next is the main result of this section.

Theorem 3.1. *Consider $(X_{A_1}, \dots, X_{A_d})$ and $(X_{B_1}, \dots, X_{B_d})$ two binomial random vectors represented as in (3.1). The conditions*

$$E[X_{A_i}] = E[X_{B_i}], \quad 1 \leq i \leq d, \quad (3.2)$$

and

$$\text{Cov}(X_{A_i}, X_{A_j}) \leq \text{Cov}(X_{B_i}, X_{B_j}), \quad 1 \leq i < j \leq d, \quad (3.3)$$

are necessary and sufficient for the supermodular ordering

$$(X_{A_1}, \dots, X_{A_d}) \leq_{\text{sm}} (X_{B_1}, \dots, X_{B_d}).$$

Proof. It is well-known, cf. e.g. Theorem 3.9.5 of [6], that for any couple (X, Y) of d -dimensional random vectors, the condition $X \leq_{\text{sm}} Y$ implies (3.2) and (3.3), therefore it suffices to show sufficiency. Using induction, it is also sufficient to consider the case where

$$\text{Cov}(X_{B_k}, X_{B_l}) = \text{Cov}(X_{A_k}, X_{A_l}) + pq, \quad (3.4)$$

for some given $1 \leq k < l \leq d$, and

$$\text{Cov}(X_{B_i}, X_{B_j}) = \text{Cov}(X_{A_i}, X_{A_j}), \quad 1 \leq i \leq j \leq d, \quad (i, j) \neq (k, l). \quad (3.5)$$

By the Möbius inversion formula (2.2) there is a unique way (up to a permutation of $\{1, \dots, n\}$) to choose $(A(e_{k,l}))_{1 \leq k \leq l \leq d}$ and $(B(e_{k,l}))_{1 \leq k \leq l \leq d}$ satisfying (3.4) and (3.5) respectively, with the relation

$$pq|B(e_{i,j})| = \sum_{e_{i,j} \preceq e_{x,y}} \mu(e_{x,y}, e_{i,j}) \text{Cov}(X_{B_x}, X_{B_y})$$

$$\begin{aligned}
&= pq \mathbf{1}_{\{e_{i,j} \preceq e_{k,l}\}} \mu(e_{k,l}, e_{i,j}) + \sum_{e_{i,j} \preceq e_{x,y}} \mu(e_{x,y}, e_{i,j}) \text{Cov}(X_{A_x}, X_{A_y}) \\
&= pq \mathbf{1}_{\{e_{i,j} \preceq e_{k,l}\}} \mu(e_{k,l}, e_{i,j}) + pq |A(e_{i,j})|, \quad 1 \leq i \leq j \leq d,
\end{aligned}$$

from (2.2), i.e.

$$|B(e_{i,j})| = \mathbf{1}_{\{e_{i,j} \preceq e_{k,l}\}} \mu(e_{k,l}, e_{i,j}) + |A(e_{i,j})|, \quad 1 \leq i \leq j \leq d. \quad (3.6)$$

Given the children $e_{k,l} \setminus \{k\}$, $e_{k,l} \setminus \{l\} \in C_d$ and grandchild $e_{k,l} \setminus \{k, l\}$ of $e_{k,l} \in C_d$, by (2.4a)-(2.4b) and (3.6) we have

$$\begin{cases} |B(e_{k,l})| = |A(e_{k,l})| + 1, \\ |B(e_{k,l} \setminus \{k\})| = |A(e_{k,l} \setminus \{k\})| - 1, \\ |B(e_{k,l} \setminus \{l\})| = |A(e_{k,l} \setminus \{l\})| - 1, \\ |B(e_{k,l} \setminus \{k, l\})| = |A(e_{k,l} \setminus \{k, l\})| + 1, \end{cases}$$

with $|B(e_{i,j})| = |A(e_{i,j})|$ in all other cases since $\mu(e_{k,l}, e_{i,j}) = 0$. We choose to realize the above as

$$\begin{cases} A(e_{k,l}) = B(e_{k,l}) \setminus \{k\}, \\ B(e_{k,l} \setminus \{k\}) = A(e_{k,l} \setminus \{k\}) \setminus \{k\}, \\ B(e_{k,l} \setminus \{l\}) = A(e_{k,l} \setminus \{l\}) \setminus \{l\}, \\ A(e_{k,l} \setminus \{k, l\}) = B(e_{k,l} \setminus \{k, l\}) \setminus \{l\}, \end{cases} \quad (3.7)$$

for some given $1 \leq k < l \leq d$, with $k, l \notin B(e_{i,j}) = A(e_{i,j})$ in all other cases. Noting that

$$l \in B(e_{k,l} \setminus \{k, l\}), \quad k \in A(e_{k,l} \setminus \{k\}), \quad l \in A(e_{k,l} \setminus \{l\}),$$

and

$$B(e_{k,l} \setminus \{k, l\}) \cap B_k = \emptyset, \quad B(e_{k,l} \setminus \{k, l\}) \cap B_l = \emptyset, \quad A(e_{k,l} \setminus \{k\}) \cap A_k = \emptyset, \quad A(e_{k,l} \setminus \{l\}) \cap A_l = \emptyset,$$

we find that

$$l \notin B_k, \quad l \notin B_l, \quad k \notin A_k, \quad l \notin A_l.$$

Hence, using the symmetric difference operator $A \setminus B := A \cap B^c$, for $i = 1, \dots, d$ we have

$$A_i = \begin{cases} (B_k \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})) \cup A(e_{k,l}) \cup \{l\}, & i = k, \\ (B_i \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})) \cup A(e_{k,l}) \cup \{k\} \cup A(e_{k,l} \setminus \{k, l\}) \cup \{l\}, & i \notin \{k, l\}, \\ (B_l \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})) \cup A(e_{k,l}) \cup \{k\}, & i = l, \end{cases} \quad (3.8)$$

and

$$B_i = \begin{cases} (B_k \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})) \cup B(e_{k,l}), & i = k, \\ (B_i \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})) \cup B(e_{k,l}) \cup B(e_{k,l} \setminus \{k, l\}), & i \notin \{k, l\}, \\ (B_l \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})) \cup B(e_{k,l}), & i = l. \end{cases} \quad (3.9)$$

In other words, from (3.7) we can write

$$\begin{cases} X_{B(e_{k,l})} = X_{A(e_{k,l})} + U, \\ X_{A(e_{k,l} \setminus \{k\})} = X_{B(e_{k,l} \setminus \{k\})} + U, \\ X_{A(e_{k,l} \setminus \{l\})} = X_{B(e_{k,l} \setminus \{l\})} + V, \\ X_{B(e_{k,l} \setminus \{k, l\})} = X_{A(e_{k,l} \setminus \{k, l\})} + V, \end{cases} \quad (3.10)$$

where $U, V \in \{Z_1, \dots, Z_n\}$ are two independent Bernoulli random variables, while we have $X_{B(e_{i,j})} = X_{A(e_{i,j})}$ in all other cases, and from (3.8)-(3.9) we get

$$X_{A_i} = \begin{cases} X_{B_k \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})} + X_{A(e_{k,l})} + V, & i = k, \\ X_{B_i \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})} + X_{A(e_{k,l})} + U + X_{A(e_{k,l} \setminus \{k, l\})} + V, & i \notin \{k, l\}, \\ X_{B_l \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})} + X_{A(e_{k,l})} + U, & i = l, \end{cases} \quad (3.11)$$

and

$$X_{B_i} = \begin{cases} X_{B_k \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})} + X_{B(e_{k,l})}, & i = k, \\ X_{B_i \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})} + X_{B(e_{k,l})} + X_{B(e_{k,l} \setminus \{k, l\})}, & i \notin \{k, l\}. \\ X_{B_l \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k, l\})} + X_{B(e_{k,l})}, & i = l. \end{cases} \quad (3.12)$$

Now, for any supermodular function $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ we have, using (3.12) and (3.10),

$$\begin{aligned}
& E \left[\phi \left((X_{B_i})_{1 \leq i \leq d} \right) \right] \\
&= E \left[\phi \left(\left(X_{B_i \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k,l\})} + X_{B(e_{k,l})} + X_{B(e_{k,l} \setminus \{k,l\})} \mathbf{1}_{\{i \notin \{k,l\}\}} \right)_{1 \leq i \leq d} \right) \right] \\
&= E \left[\phi \left(\left(X_{B_i \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k,l\})} + X_{A(e_{k,l})} + U + (X_{A(e_{k,l} \setminus \{k,l\})} + V) \mathbf{1}_{\{i \notin \{k,l\}\}} \right)_{1 \leq i \leq d} \right) \right] \\
&\geq E \left[\phi \left(\left(X_{B_i \setminus B(e_{k,l}) \setminus B(e_{k,l} \setminus \{k,l\})} + X_{A(e_{k,l})} + U \mathbf{1}_{\{i \neq k\}} + X_{A(e_{k,l} \setminus \{k,l\})} \mathbf{1}_{\{i \notin \{k,l\}\}} + V \mathbf{1}_{\{i \neq l\}} \right)_{1 \leq i \leq d} \right) \right] \\
&= E \left[\phi \left((X_{A_i})_{1 \leq i \leq d} \right) \right],
\end{aligned}$$

where we used (3.11) for the last equality. As for the above inequality, it follows from

$$\begin{aligned}
& E \left[\phi (U, U + V, \dots, U + V, U) \right] \\
&= p^2 \phi(1, 2, \dots, 2, 1) + q^2 \phi(0, 0, \dots, 0, 0) + pq \phi(1, 1, \dots, 1, 1) + pq \phi(0, 1, \dots, 1, 0) \\
&\geq p^2 \phi(1, 2, \dots, 2, 1) + q^2 \phi(0, 0, \dots, 0, 0) + pq \phi(1, 1, \dots, 1, 0) + pq \phi(0, 1, \dots, 1, 1) \\
&= E \left[\phi (U, U + V, \dots, U + V, V) \right],
\end{aligned}$$

for all supermodular functions $\phi : \mathbb{R}^{|e_{k,l}|} \longrightarrow \mathbb{R}$, where $|e_{k,l}|$ denotes the cardinality of $e_{k,l}$ whose indices are arranged as $\{k, \dots, l\}$ for convenience of notation, and we did not consider indices $j \notin e_{k,l}$ as U and V do not belong to X_j in this case. \square

Multivariate Gaussian vectors

From the central limit theorem, Theorem 3.1 can be used to deal with centered multivariate Gaussian random vectors (U_1, \dots, U_d) and (V_1, \dots, V_d) with covariance matrices

$$(\text{Cov}(U_i, U_j))_{1 \leq i \leq j \leq d} \quad \text{and} \quad (\text{Cov}(V_i, V_j))_{1 \leq i \leq j \leq d}.$$

In this case we can apply the Möbius inversion (2.2) in order to determine the variance coefficients $(\sigma_{k,l}^2)_{1 \leq k \leq l \leq d}$ and $(\eta_{k,l}^2)_{1 \leq k \leq l \leq d}$ in the decomposition (2.5). Those coefficients can then be obtained as the respective limits of variances $(\text{Var}[X_{k,l}^n / \sqrt{n}])_{1 \leq k \leq l \leq d}$ and $(\text{Var}[Y_{k,l}^n / \sqrt{n}])_{1 \leq k \leq l \leq d}$ of independent binomial random variables such that

$$U_i^n := \frac{1}{\sqrt{n}} \sum_{i \in e_{k,l}} (X_{k,l}^n - E[X_{k,l}^n]) \quad \text{and} \quad V_i^n := \frac{1}{\sqrt{n}} \sum_{i \in e_{k,l}} (Y_{k,l}^n - E[Y_{k,l}^n]), \quad i = 1, \dots, d,$$

converge in distribution to (U_1, \dots, U_d) and (V_1, \dots, V_d) respectively. Letting n tend to infinity in $\text{Cov}(U_i^n, U_j^n) \leq \text{Cov}(V_i^n, V_j^n)$, the condition $\text{Cov}(U_i, U_j) \leq \text{Cov}(V_i, V_j)$

of Theorem 3.1, $1 \leq i < j \leq d$, becomes necessary and sufficient for $(U_1, \dots, U_d) \leq_{\text{sm}} (V_1, \dots, V_d)$ to hold. In this way we recover the result of [5], Theorem 4.2, for Gaussian random vectors, cf. also Theorem 3.13.5 of [6].

A similar argument results from Theorem 4.2 below in the Poisson case, using the convergence in distribution from renormalized binomial random variables to Poisson random variables. In the next section we provide a proof of such a result using Lévy measures for infinitely divisible Poisson random vectors.

4 Poisson random vectors

Recall that any d -dimensional infinitely divisible Poisson random vector $X = (X_1, \dots, X_d)$ is defined by its characteristic function

$$E[e^{i\langle \bar{t}, X \rangle}] = \exp \left(\int_{\mathbb{R}^d} (e^{i\langle \bar{t}, x \rangle} - 1) \mu(dx) \right),$$

where $\bar{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^d , and the Lévy measure

$$\mu(dx) := \sum_{\emptyset \neq S \subset \{1, 2, \dots, d\}} a_S \delta_{e_S}(dx),$$

is supported on C_d , where $\delta_{e_S}(dx)$ denotes the Dirac measure at the point $e_S \in C_d$, and $(a_S)_{\emptyset \neq S \subset \{1, 2, \dots, d\}}$ is a family of nonnegative coefficients.

Equivalently, $X = (X_1, \dots, X_d)$ can be represented as

$$X_i = \sum_{\substack{S \in \{0, 1\}^d \\ S \neq \emptyset}} \mathbf{1}_{\{i \in S\}} X_S = \sum_{\substack{S \subset \{1, 2, \dots, d\} \\ S \ni i}} X_S, \quad i = 1, \dots, d,$$

where $(X_S)_{\emptyset \neq S \subset \{1, 2, \dots, d\}}$ is a family of $2^d - 1$ independent Poisson random variables with respective intensities $(a_S)_{\emptyset \neq S \subset \{1, 2, \dots, d\}}$, cf. also Theorem 3 of [3].

In order to characterize the ordering of Poisson random vectors based on the data of their covariance matrices which contain only $d(d+1)/2$ components, we restrict ourselves to Lévy measures of the form

$$\mu(dx) = \sum_{1 \leq k \leq l \leq d} a_{k,l} \delta_{e_{k,l}}(dx), \quad (4.1)$$

on C_d , where $a_{k,l} \in \mathbb{R}_+$, $1 \leq k \leq l \leq d$. In other words, we have

$$X = \sum_{i=1}^d e_i \sum_{i \in e_{k,l}} X_{k,l} = \sum_{1 \leq k \leq l \leq d} X_{k,l} e_{k,l}, \quad (4.2)$$

where $(X_{k,l})_{1 \leq k \leq l \leq d}$ is a family of independent Poisson random variables whose respective intensity parameters $(a_{i,j})_{1 \leq i \leq j \leq d}$ satisfy $\text{Var}[X_{k,l}] = E[X_{k,l}] = a_{k,l}$, $1 \leq k \leq l \leq d$, with the inversion formula

$$a_{k,l} = \sum_{e_{k,l} \preceq e_{i,j}} \mu(e_{i,j}, e_{k,l}) \text{Cov}(X_i, X_j), \quad 1 \leq k \leq l \leq d, \quad (4.3)$$

that follows from (2.2).

Supermodular ordering of Poisson random vectors

Theorem 4.2 below is a direct consequence of the following Lemma 4.1 which provides the decomposition

$$\mu(dx) = \sum_{i=1}^d \text{Var}[X_i] \delta_{e_i} + \sum_{1 \leq i < j \leq d} \text{Cov}(X_i, X_j) (\delta_{e_{i,j}} + \delta_{e_{i,j} \setminus \{i,j\}} - \delta_{e_{i,j} \setminus \{i\}} - \delta_{e_{i,j} \setminus \{j\}})$$

of the Lévy measure $\mu(dx)$ on $C_d \setminus \{0\}$, using the covariance matrix of $(X_i)_{i=1,\dots,d}$.

Lemma 4.1. *For any function $\phi : C_d \longrightarrow \mathbb{R}$ such that $\phi(0) = 0$ we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x) \mu(dx) &= \sum_{i=1}^d E[X_i] \phi(e_i) \\ &+ \sum_{1 \leq i < j \leq d} \text{Cov}(X_i, X_j) (\phi(e_{i,j}) + \phi(e_{i,j} \setminus \{i,j\}) - \phi(e_{i,j} \setminus \{i\}) - \phi(e_{i,j} \setminus \{j\})). \end{aligned}$$

Proof. By the Möbius inversion formula (2.2) we have

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(x) \mu(dx) &= \sum_{1 \leq k \leq l \leq d} a_{k,l} \phi(e_{k,l}) \\ &= \sum_{1 \leq k \leq l \leq d} \phi(e_{k,l}) \sum_{e_{k,l} \preceq e_{i,j}} \mu(e_{i,j}, e_{k,l}) \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^d \text{Cov}(X_i, X_i) \sum_{e_k \preceq e_i} \mu(e_i, e_k) \phi(e_k) + \sum_{1 \leq i < j \leq d} \text{Cov}(X_i, X_j) \sum_{\substack{e_{k,l} \preceq e_{i,j} \\ 1 \leq k < l \leq d}} \mu(e_{i,j}, e_{k,l}) \phi(e_{k,l}) \end{aligned}$$

$$= \sum_{i=1}^d E[X_i] \phi(e_i) + \sum_{1 \leq i < j \leq d} \text{Cov}(X_i, X_j) (\phi(e_{i,j}) + \phi(e_{i,j} \setminus \{i, j\}) - \phi(e_{i,j} \setminus \{i\}) - \phi(e_{i,j} \setminus \{j\})),$$

where we used (2.4a)-(2.4b) and the fact that $e_k \preceq e_i$ if and only if $k = i$. \square

Consider now two Poisson random vectors X and Y whose respective Lévy measures μ and ν are represented as

$$\mu(dx) = \sum_{1 \leq i \leq j \leq d} a_{i,j} \delta_{e_{i,j}}(dx) \quad \text{and} \quad \nu(dx) = \sum_{1 \leq i \leq j \leq d} b_{i,j} \delta_{e_{i,j}}(dx),$$

as in (4.1). If X_i has the same distribution as Y_i for all $i = 1, \dots, d$ then $E[X_i] = E[Y_i]$, $i = 1, \dots, d$, and Lemma 4.1 shows that

$$\begin{aligned} & \int_{\mathbb{R}^d} \phi(y) \nu(dy) - \int_{\mathbb{R}^d} \phi(x) \mu(dx) \\ &= \sum_{1 \leq i < j \leq d} (\text{Cov}(Y_i, Y_j) - \text{Cov}(X_i, X_j)) (\phi(e_{i,j}) + \phi(e_{i,j} \setminus \{i, j\}) - \phi(e_{i,j} \setminus \{i\}) - \phi(e_{i,j} \setminus \{j\})). \end{aligned} \quad (4.4)$$

Relation (4.4) shows in particular that the nonnegativity of the coefficients

$$\text{Cov}(Y_i, Y_j) - \text{Cov}(X_i, X_j) \geq 0, \quad 1 \leq i < j \leq d, \quad (4.5)$$

becomes a necessary and sufficient condition for the supermodular ordering of the (finite support) Lévy measures μ and ν .

The next Theorem 4.2 reformulates (4.5) as a necessary and sufficient condition for supermodular ordering of infinitely divisible Poisson random vector, based on Theorem 4.5 of [1] which allows us to carry over the notion of supermodularity from the finite support setting of Lévy measures μ, ν on the cube C_d , to the infinite support setting of Poisson random variables.

Theorem 4.2. *Consider two Poisson random vectors X and Y both represented as in (4.2). Then the conditions*

$$E[X_i] = E[Y_i], \quad 1 \leq i \leq d,$$

and

$$\text{Cov}(X_i, X_j) \leq \text{Cov}(Y_i, Y_j), \quad 1 \leq i < j \leq d, \quad (4.6)$$

are necessary and sufficient for the supermodular ordering $X \leq_{\text{sm}} Y$.

Proof. By Theorem 4.5 in [1] it suffices to show that

$$\int_{\mathbb{R}^d} \phi(x) \mu(dx) \leq \int_{\mathbb{R}^d} \phi(y) \nu(dy) \quad (4.7)$$

for all supermodular functions $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$, where $\mu(dx)$ and $\nu(dy)$ respectively denote the Lévy measures of X and Y . By Lemma 4.1 we have the identity

$$\begin{aligned} & \int_{\mathbb{R}^d} \phi(y) \nu(dy) - \int_{\mathbb{R}^d} \phi(x) \mu(dx) \\ &= \sum_{1 \leq i < j \leq d} (\text{Cov}(Y_i, Y_j) - \text{Cov}(X_i, X_j)) (\phi(e_{i,j}) + \phi(e_{i,j} \setminus \{i, j\}) - \phi(e_{i,j} \setminus \{i\}) - \phi(e_{i,j} \setminus \{j\})) \end{aligned}$$

under condition (4.6), which allows us to conclude to (4.7) for all supermodular functions ϕ . \square

The next proposition is obtained as in Proposition 4.3 of [4] by extending Theorem 4.5 of [1] to nondecreasing supermodular functions ϕ on \mathbb{R}^d satisfying $\phi(0) = 0$, using the same approximation as in Lemma 4.4 therein.

Proposition 4.3. *Consider two Poisson random vectors X and Y both represented as in (4.2), and assume that*

$$E[X_i] \leq E[Y_i], \quad 1 \leq i \leq d,$$

and

$$\text{Cov}(X_i, X_j) \leq \text{Cov}(Y_i, Y_j), \quad 1 \leq i < j \leq d.$$

Then we have

$$E[\Phi(X)] \leq E[\Phi(Y)]$$

for all nondecreasing supermodular functions $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$.

Sums of binomial, Gaussian and Poisson vectors

By Theorem 4.2 of [5] on Gaussian random vectors, Theorems 3.1 and 4.2 above, and the fact that the supermodular ordering is closed under convolution, cf. Theorem 3.9.14-(C) of [6], deduce that the supermodular ordering of a sum of independent binomial, Gaussian and Poisson vectors, is implied by the componentwise ordering of their respective covariances. Proposition 4.3 admits an analog extension to sums of binomial, Gaussian and Poisson random vectors.

Convex ordering

Proposition 4.4. *Consider two Poisson random vectors X and Y both represented as in (4.2). Then we have $X \leq_{\text{cx}} Y$ if and only if X and Y have same distributions.*

Proof. Assume $X \leq_{\text{cx}} Y$, i.e. we have

$$E[\Phi(X)] \leq E[\Phi(Y)]$$

for all *convex* functions $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$. Clearly, this implies $E[X_k] = E[Y_k]$, $k = 1, \dots, d$. Next, choosing any $1 \leq k < l \leq d$ we check that the function

$$(x_1, \dots, x_d) \mapsto \phi_{k,l}(x_1, \dots, x_d) := \max \left(0, x_l - x_k - \sum_{a \notin e_{k,l}} x_a \right)$$

is convex on \mathbb{R}^d , with $\phi_{k,l}(e_{i,j}) = 1$ when $e_{i,j}$ is a (non-strict) descendant of $e_{k,l} \setminus \{k\}$ that contains l , and $\phi_{k,l}(e_{i,j}) = 0$ in all other cases. This yields

$$\phi_{k,l}(e_{k,l}) + \phi_{k,l}(e_{k,l} \setminus \{k, l\}) - \phi_{k,l}(e_{k,l} \setminus \{k\}) - \phi_{k,l}(e_{k,l} \setminus \{l\}) = -1,$$

and

$$\phi_{k,l}(e_{i,j}) + \phi_{k,l}(e_{i,j} \setminus \{i, j\}) - \phi_{k,l}(e_{i,j} \setminus \{i\}) - \phi_{k,l}(e_{i,j} \setminus \{j\}) = 0$$

when $(i, j) \neq (k, l)$. Hence by Lemma 4.1, the condition $\text{Cov}(Y_k, Y_l) > \text{Cov}(X_k, X_l)$ would imply

$$\begin{aligned} & \int_{\mathbb{R}^d} \phi(y) \nu(dy) - \int_{\mathbb{R}^d} \phi(x) \mu(dx) \\ &= \sum_{1 \leq i < j \leq d} (\text{Cov}(Y_i, Y_j) - \text{Cov}(X_i, X_j)) (\phi(e_{i,j}) + \phi(e_{i,j} \setminus \{i, j\}) - \phi(e_{i,j} \setminus \{i\}) - \phi(e_{i,j} \setminus \{j\})) \\ &= (\text{Cov}(Y_k, Y_l) - \text{Cov}(X_k, X_l)) (\phi(e_{k,l}) + \phi(e_{k,l} \setminus \{k, l\}) - \phi(e_{k,l} \setminus \{k\}) - \phi(e_{k,l} \setminus \{l\})) \\ &< 0, \end{aligned}$$

which would contradict $X \leq_{\text{cx}} Y$ by the same argument as in part (b) of the proof of Theorem 4.5 in [1]. Hence we have $\text{Cov}(Y_k, Y_l) \leq \text{Cov}(X_k, X_l)$, and proceeding similarly by exchanging k and l with the convex function

$$(x_1, \dots, x_d) \mapsto \phi_{k,l}(x_1, \dots, x_d) := \max \left(0, x_k - x_l + \sum_{a \notin e_{k,l}} x_a \right)$$

we deduce that $\text{Cov}(Y_k, Y_l) = \text{Cov}(X_k, X_l)$ for all $1 \leq k < l \leq d$, hence X and Y have same distribution from (4.3). \square

References

- [1] N. Bäuerle, A. Müller, and A. Blatter. Dependence properties and comparison results for Lévy processes. *Math. Meth. Operat. Res.*, 67:161–186, 2008.
- [2] T. Hu, C. Xie, and L. Ruan. Dependence structures of multivariate Bernoulli random vectors. *J. Multivariate Anal.*, 94(1):172–195, 2005.
- [3] K. Kawamura. The structure of multivariate Poisson distribution. *Kodai Math. J.*, 2(3):337–345, 1979.
- [4] B. Kızıldemir and N. Privault. Supermodular ordering of Poisson arrays. *Statist. Probab. Lett.*, 98:136–143, 2015.
- [5] A. Müller and M. Scarsini. Some remarks on the supermodular order. *J. Multivariate Anal.*, 73:107–119, 2000.
- [6] A. Müller and D. Stoyan. *Comparison methods for stochastic models and risks*. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester, 2002.
- [7] G. Peccati and M. Taqqu. *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with Computer Implementation*. Bocconi & Springer Series. Springer, 2011.
- [8] G.-C. Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:340–368, 1964.